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Mutations of steady cellular flows in the Taylor experiment

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Aspects of the various steady states of Taylor-vortex flow between concentric cylinders have been investigated by means of flow visualization. The experiments have focused principally on *the evolution of the primary flow*, that is, on the continuum of steady states parametrized by the Reynolds number R, beginning at small R where the primary flow is the only one possible. For any particular aspect ratio Γ, the primary flow develops a well-defined pattern of cells at higher R, but then other steady cellular flows (secondary modes) are also possible. The observations presented demonstrate mutations of the primary flow as Γ is varied through critical values: its R-dependent evolution is thereby switched from one to another array of cells realized at higher R. In each of four cases (4–6, 6–8, 8–10 and 10–12 cells), the mutation is shown to involve hysteresis of the primary-flow locus and complicated interactions with secondary modes.

Following a description of the apparatus in §2, a discussion of the experimental method used to observe the often delicate hysteresis effects is given in §3. The experimental results in §4 are in broad agreement with abstract mathematical ideas that have been previously shown to bear on the Taylor experiments, but several new and surprising features, such as the coupling between pairs of cells, have been uncovered.

1. Introduction

The aim of this experimental study was to extend the findings of Benjamin (1978*a*, *b*) about Taylor–Couette flow in an annulus of comparatively small length with the inner cylindrical wall rotating. In particular, further information has been obtained concerning mutations of the *primary flow* as the length of the annulus is changed. For any given proportions of the annulus, the primary flow is uniquely definable as the steady flow realized by very gradual increases in the angular speed Ω of the inner cylinder from small values. The main dynamical parameter in these experiments is the Reynolds number defined by $R = \Omega r_1 d/\nu$, where r_1 is the radius of the inner cylinder, $d = r_2 - r_1$ the width of the fluid-filled annulus and ν the kinematic viscosity of the fluid. The two geometrical parameters are the radius ratio $\eta = r_1/r_2$ and the aspect ratio $\Gamma = l/d$, where l is the length of the annulus. In the present experiments, as in Benjamin's, the ends of the annulus are fixed solid planes perpendicular to the axis of the concentric cylinders.

At small values of R the observed steady motion is mainly in circles, but some three-dimensional features may appear near the fixed end walls. As R is increased into a narrow, quasi-critical range, Taylor cells grow progressively from the ends, fitting together at the centre to form the primary cellular flow that is uniquely determined

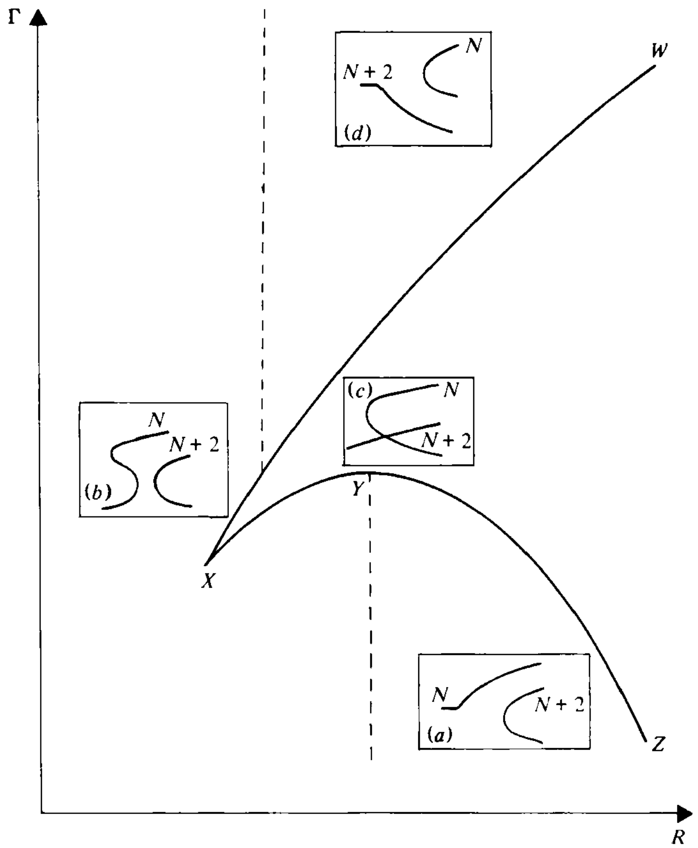

FIGURE 1. Schematic diagram of exchange procedure in (R, Γ)-plane. Inset figures show the relevant state diagrams in the (R, f)-plane. In this example they are ordered (a) → (d) as Γ increases.

over the respective range of Γ. The cellular mode that will become the primary flow in the next range of Γ is also realizable as a stable steady flow at sufficiently high values of R, and it may be produced in practice when the flow settles down after a sudden start of the rotation at some value substantially above the quasi-critical range. Realized in these circumstances, the flow is called a *secondary* mode. It evidently cannot survive indefinitely as R is gradually reduced, and it collapses catastrophically at a well-defined critical value of R.

The form of the selection process, as represented by a graph of such critical values in the (Γ, R)-plane, has been predicted generally in the theoretical work of Benjamin (1978*a*) and Schaeffer (1980). In particular, the change-over of the primary mode from one array of cells to another (e.g. 4 cells giving way to 6) is accompanied by a hysteresis, indicated by a cusp pointing obliquely to the R-axis in the (Γ, R)-chart. However, the exact orientation of this cusp, whether pointed up or down relative to the R-axis, is not predicted by the theory.

The schematic (Γ, R)-diagram shown in figure 1 illustrates the exchange process. In the range of Γ below Y the N-cell mode is primary and develops smoothly with gradual increase in R, as is shown in state diagram (a). The ($N + 2$)-cell state can exist as a secondary mode to the right of YZ, collapsing along this line, on reduction of R, to the primary N-cell mode. An increase in Γ leads to the development of a fold in the primary locus of the state diagram (*b*), which is manifested as the cusped region XY in the (R, Γ)-plane. The two state curves then became linked (diagram (*c*)) at Y, which is thus the bifurcation point for the primary flow. In the Γ-range WX the roles are reversed (diagram (*d*)), and the $(N+2)$-cell mode is now primary. Thus one primary locus is *continuously* deformed into another by the variation of Γ. A cusp pointing in the opposite direction indicates that the exchange proceeds as $d \rightarrow c \rightarrow b \rightarrow a$ as Γ is increased.

Benjamin (1978*b*) exemplified the general form of the change-over process in his experiments on the mutation from a two-cell to a four-cell primary flow. This first example of the change-over process may be judged highly special, owing to the close proximity of the end walls in this case, and so it is plainly desirable to investigate other examples where end effects are less conspicuous.

It should be emphasized that the hysteresis and other phenomena involved in the process of mutation of the primary flow depend very delicately on Γ and R. Consequently, great care is needed to observe them definitely, and they appear to have been missed by various investigators in the past (e.g. see the review article by Di Prima & Swinney (1981) for a survey of the many experimental studies of Taylor flow, particularly § 6.6 for a discussion of 'end effects'). For example, in a study of Taylor–Couette flow with greater numbers of cells, Cole (1976) recorded what amounts to a succession of different cellular arrays taking priority as the primary flow, but he did not observe any hysteresis effects, nor evidence of secondary-mode behaviour. This absence of accord with the present findings may be largely accountable to the fact that in his experiments the upper boundary of the fluid was a free surface, whose effects may be estimated to swamp those now principally in question.

2. Experimental apparatus

The design of the apparatus was an elaboration of that used in the previous study (Benjamin 1978*b*). The fixed outer cylinder is a glass tube of precision bore having inner radius $31{\cdot}75 \pm 0{\cdot}02$ mm, and the inner rotating cylinder is machined aluminium of radius $19{\cdot}05 \pm 0{\cdot}01$ mm. The radius ratio is thus 0·60, and the gap width $d = 12{\cdot}70$ mm.

The upper surface of the annular section filled with fluid is the bottom of a PTFE collar, the position of which is continuously adjustable. Posts supporting the collar are rigidly attached to an overhead gantry, and thereby the vertical position of the collar can be varied over a wide range. Very fine adjustments of position are affected by a calibrated micrometer lead screw incorporated into the supporting device. The available range of aspect ratio is 0–16.

The inner and outer cylinders are centred in a square glass box and based upon a thick plate of Perspex. The surrounding jacket is filled with liquid paraffin to minimize optical distortion due to the curved surfaces. The jacket has the additional advantage of buffering small temperature fluctuations in the ambient air of the temperature-controlled cabinet which encloses the whole apparatus. Temperature variations are thus limited to less than 0·1 °C, but as a check a complete record of temperature in the paraffin-filled jacket was kept in the experiments.

Glycerol–water mixtures of varying concentrations were used, whose viscosities (typically $\nu = 5\,\mathrm{cS}$) were measured, at the end of each series of experiments, by a suspended-level viscometer. As in previous studies, the flow was visualized by means of Mearlmaid AA pearlessence added to the liquid, which was illuminated with a slit of light in a plane at right angles to the observer. Cellular interfaces could be seen distinctly as narrow dark bands across the field of view. Thus cell numbers were always obvious, and cell heights could be readily measured by use of a cathetometer external to the cabinet. This instrument was also used to measure the length of the annular section, and a check was thereby made against the micrometer measurements when fine adjustments in length were required.

The inner cylinder was mounted between an adjustable end-bearing at the bottom and a bush support in the lid closing the apparatus. The rotor was driven by a stepping motor through a reduction gear and belt drive. The rotor speed was maintained to within the frequency stability of the oscillator that controlled the stepping motor (0·01 %), and its value was displayed digitally by an electronic counter. In addition, calibration runs were made over the speed ranges used in order to check that belt slippage was insignificant.

3. Experimental method

The measurements illuminating the changing roles of various cellular flows as primary and secondary modes were all made in the same way, which will now be explained. As already noted, for a respective range of Γ, a particular mode with an even number of cells occurs as the unique primary flow, and accordingly it can be realized by increasing the speed of the inner cylinder gradually from rest. The cellular structure becomes quite distinct at values of R somewhat above the quasi-critical range where Taylor cells first appear. Cellular flows are also realizable as secondary modes, at values of Γ for which another flow is primary. These flows, which are stable only for large enough R, can be obtained in practice as the outcome of sudden starts of the apparatus in certain narrow speed ranges (Benjamin 1978*b*; Benjamin & Mullin 1981).

Upon gradual reduction of speed, the secondary flows eventually lose stability and collapse at some critical value of Reynolds number. Following the transient motion then generated, a steady state is regained in the mode that is the primary flow for the respective Γ in the ranges of R studied here. (Other collapse routes found at much higher values of R are reported in detail in Benjamin & Mullin (1981).) When Γ is far removed from any range wherein the change-over in the primary mode takes place, the collapses of secondary modes are quite distinct events, and critical values of R are easy to estimate accurately. As a change-over range of Γ is approached, they become much less sharp, but with care they are still observable.

Observations are begun in ranges of Γ well above and below the expected change-over range; and the flow with N or $N \pm 2$ cells, where N is an even integer, is established as a secondary mode. The rotor speed is then reduced in small steps, with adequate settling times allowed between changes, and the speed at which collapse finally occurs is determined. Loci of critical speeds (R_c, Γ) are thus gradually built up. These two experimental curves point to the change-over region of Γ, for which a new approach has to be adopted.

In this range R is *increased* from low values that lie below the exchange region, in small discrete steps allowing adequate settling time between increments. The cells develop from the ends, but fail to mesh together in the central region which appears to consist of stagnant fluid. In these speed ranges it is difficult to judge whether this is an N or $N+2$ mode.

As the speed is further increased, one observes either the sudden appearance of a pair of cells in the stagnant region or the sudden disappearance of the region altogether. Then there exists a definite set of cells along the entire length of the cylinders. Upon reduction of speed the structure reverts to its original form abruptly at a lower speed. Thus a definite hysteresis is measured.

In ranges of Γ where the hysteresis is small, the changes become weakened and it becomes much more difficult to make objective estimates of critical values. However, the general trend is clear, and after many hours of observation a complete picture of the event can be recognized with some degree of confidence.

It deserves emphasis that these are extremely delicate experiments. Changes in speed when approaching critical values were never greater than 1 % (typically 0·25–0·5 %), and changes in length were even more finely controlled (to 0·02 mm). Settling times between changes were varied, between 5 min for the smaller values of Γ to half an hour for the larger ones, before consistent results could be obtained. These settling times are up to five times greater than those indicated as necessary by the empirical formula of Snyder (1969). The settling time for each bifurcation set was established at one particular value of Γ above the expected exchange region. The time between changes in speed was extended until a consistent value of R_c was obtained, and this rate of change was then used for the rest of the measurements in the set.

The experiments were thus extremely time-consuming, typically taking 15–20 days to complete one graph, of which at least 2–3 days were spent on the change-over region alone. In addition, no set of points was obtained completely using one charge of fluid as a further check on repeatability. However, it must be recognized that the sensitivity of these effects to minute changes in length may imply their absolute accuracy to be subject to small imperfections in the apparatus.

4. Results

The observations are reported as four 'bifurcation diagrams', namely sets of critical values (R, Γ) at which the observed flows lose stability and abrupt changes in the cellular structure occur. They cover the change-over processes where flows with 4, 6, 8, 10 and 12 cells successively take priority as the primary flow when the aspect ratio is varied. These sets of results will first be discussed separately, since each contains details of interest in themselves. The more general features of the results will be discussed finally in § 5, when the four experimental diagrams will be compared and discussed in relation to previous studies.

4.1. *The four-cell to six-cell transition*

The results for this case are presented in figure 2. For parameter values (R, Γ) to the right of the line AB in the figure the four-cell flow is realizable as a stable secondary mode by a sudden start of the apparatus. On gradual reductions in R, the mode

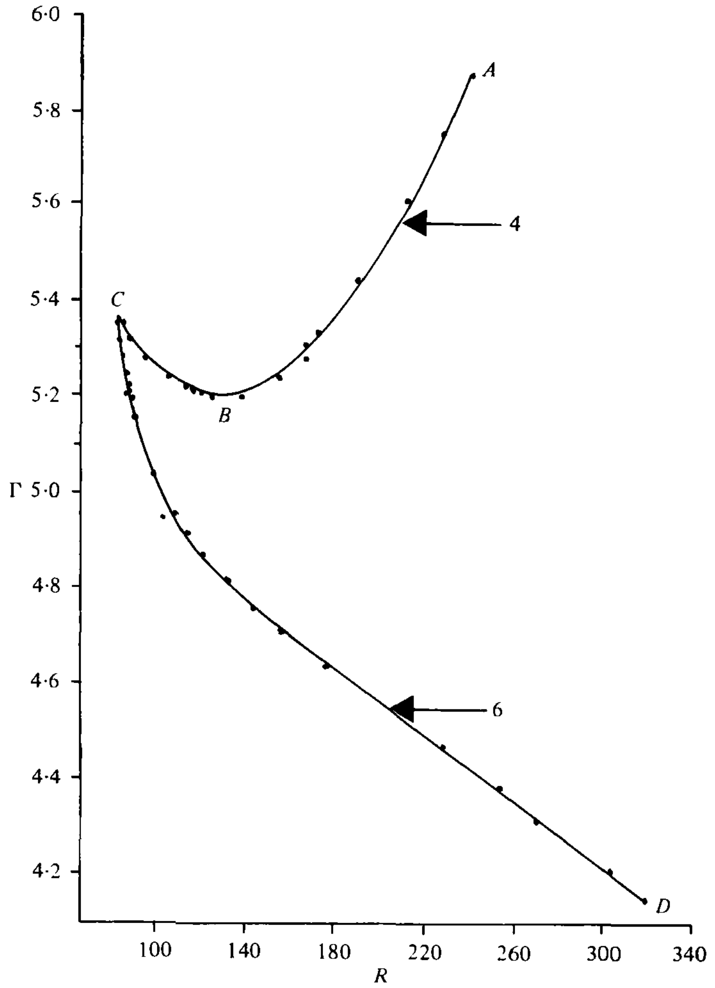

FIGURE 2. Experimentally determined bifurcation set in the (R, Γ)-plane for four-cell and six-cell flows. The arrows emphasize that the respective loci are the lower limits of stability for the secondary modes whose number of cells are indicated.

collapses when the line AB is reached, and the system settles into the six-cell mode, which is the primary flow in this range of Γ.

Similarly, the line CD is the locus of terminal states for the six-cell flow occurring as a secondary mode. Here the four-cell mode is the primary flow and is realized upon collapse of the secondary mode. The cusp-shaped region bounded to the right by BC is where the priority of the modes of the primary flow is exchanged. In this range of aspect ratio, a gradual increase of speed from a point to the left of CD will result in the initial four-cell form to jump into the six-cell one as the line BC is reached. If the speed is then gradually reduced, the six-cell mode will collapse back to the four-cell at a lower speed along CD, showing a definite hysteresis. In ranges where the hysteresis is large the jumps are very distinct, gradually becoming less clear as the aspect ratio is increased.

In a comparison of these results with those of Benjamin (1978*b*) for the mutation from 2 to 4 cells, salient differences are recognizable. First, the present cusp is directed upwards, whereas that for the 2/4 mutation is downward. Also, this cusp is much greater in width. The widest hysteresis involves a 28 % change in speed, compared with 3·6 % for the 2/4 cusp. This difference was surprising, since it had been expected that the hysteresis effects would diminish greatly with increase in the number of cells involved. However, the range of Γ spanned by the cusp is closely comparable to that measured by Benjamin, being approximately 2 mm (i.e. about 3 % of the column length). Also, the main qualitative features of this cusp have been confirmed in a parallel study (Mullin, Pfister & Lorenzen 1982) using different apparatus.

4.2. *The six-cell to eight-cell transition*

The results for this exchange are shown in figure 3. Here the line HI is the lower limit of stability for the eight-cell mode as a secondary flow, and this line is determined with comparative ease. On the other hand, the six-cell mode exhibited some peculiar features as a secondary flow, to be described presently. Along the line FG, however, it behaved in a manner more or less the same as the four-cell and other secondary flows at collapse.

In the aspect-ratio range above F it proved impossible to produce six cells as a secondary mode by the usual means of sudden starts of the rotor. An alternative method was to produce it at a shorter length and then gradually raise the upper collar, but this also proved to be difficult. This method was effective only if the rotor speed was over three times the expected collapse value obtained by an extrapolation of FG.

By use of the alternative method the upper part of EF was relatively easy to obtain, and a striking feature of the collapse was the appearance of the extra pair of cells *asymmetrically*. The new pair was formed next to either the top or bottom pair and not in the centre of the section as along FG. The lower portion of EF was very difficult to obtain as a 0·37 % change in length produced a 62·5 % change in critical speed. Nevertheless, repetition of the measurements with different inner cylinders machined to nominally the same radius produced the same form of events, and thus they are reported here with a degree of confidence.

The cusped region to the left of GH is only about 1 mm long and spans a speed range of approximately 4 % which is in accord with the 2/4-cell mutation results of Benjamin (1978*a*, *b*). Thus, measurements in this region involve exceptionally delicate adjustments of speed with very long settling times between changes before repeatable measurements can be obtained.

As in the case of the 4/6-cell mutation, the cusp is directed upwards but its point is now 5·5 % above the critical Reynolds number according to the idealized model. Around G there is no evidence of a rounding off in the loci of critical values, as had been expected, and within experimental error the lines HG and FG appeared to intersect at a sharp angle. The intersection occurs at aspect ratio 7·5, which reverses the trend set in the 4/6-cell mutation towards critical values of Γ proportional to $N+1$, where N is the lower cell number involved.

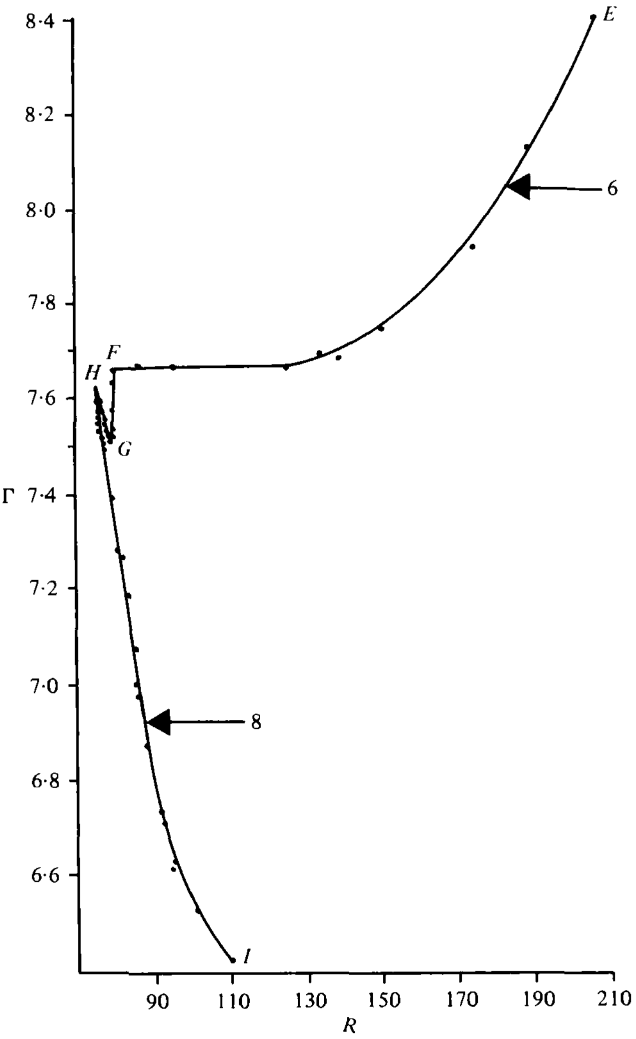

FIGURE 3. Experimentally determined bifurcation set in the (R, Γ)-plane for six-cell and eight-cell flows. The arrows emphasize that the respective loci are the lower limits of stability for the secondary modes whose number of cells are indicated.

4.3. *The eight-cell to ten-cell transition*

The study of this mutation was concentrated around the change-over region, since settling times became almost prohibitively long. The results are shown in figure 4, where as before the lines now labelled LM and JK are respectively the lower stability limits of the ten- and eight-cell secondary modes. Towards the right-hand ends of these experimental curves the collapses are distinct, but at the other ends, coinciding with the exchange region, the observations demanded exceptional care.

The largest hysteresis measured corresponds to a 2 % change in speed, and the entire length of the cusp is covered by about 0·6 mm in a total length of 120 mm. Thus the micrometer adjustment was essential to reproducible measurements.

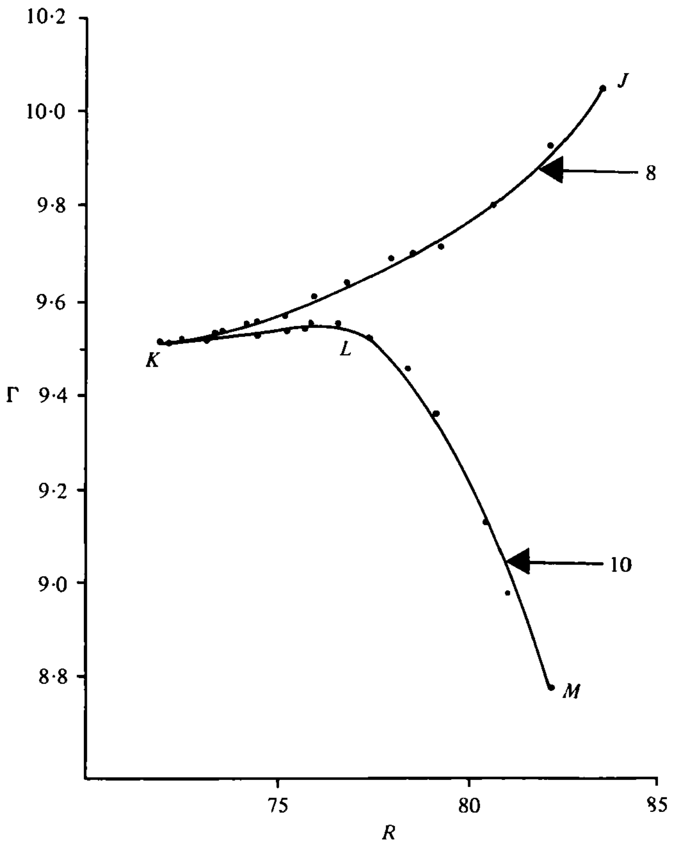

FIGURE 4. Experimentally determined bifurcation set in the (R, Γ)-plane for eight-cell and ten-cell flows. The arrows emphasize that the respective loci are the lower limits of stability for the secondary modes whose number of cells are indicated.

As the speed was increased from a point to the left of JK in the exchange region, the cells appeared to grow from the ends of the apparatus but failed to mesh properly at the centre. The flow had the appearance of eight full cells plus a stagnant region of approximately the same size as one of the developed cells. Upon further increase of speed such as to cross the line KL, the stagnation region disappeared and a distinct eight-cell structure was produced. Reduction of the speed led to the re-emergence of the stagnant region at a critical value slightly less than on KL. A small but definite hysteresis was thus found.

Unlike any other measured in this series of experiments this cusp points downwards. Thus the direction of the cusp does not appear to be predictable in any obvious way.

The point of the cusp is located at a Reynolds number just below the classical critical value, in agreement with the finding for the 2/4-cell exchange. The loci of critical values is again rounded around the bifurcation point which is located at aspect ratio approximately equal to 9·5 which is consistent with the other results.

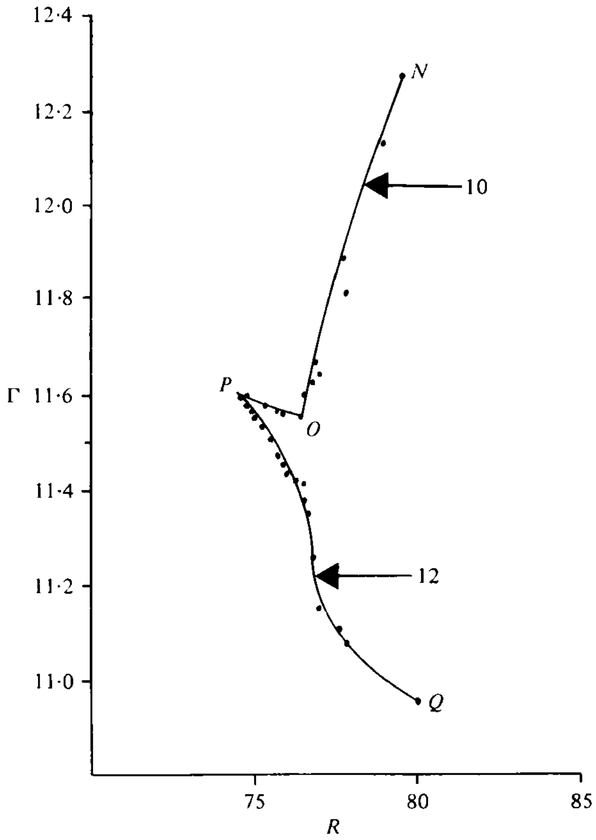

FIGURE 5. Experimentally determined bifurcation set in the (R, Γ)-plane for ten-cell and twelve-cell flows. The arrows emphasize that the respective loci are the lower limits of stability for the secondary modes whose number of cells are indicated.

4.4. *The ten-cell to twelve-cell transition*

The measurements of critical values for this exchange are shown in figure 5. The lower stability limits of the ten-cell and twelve-cell flows occurring as secondary modes are set along the lines NO and PQ.

The height of the cusped region (i.e. the difference in Γ values at points P and O) corresponds to approximately 0·6 mm in 146 mm, and the cusp spans around 2·5 % in R. From inside the cusp an increase in speed across the line PO brings about the formation of an extra pair of cells in the central region of the flow. Reduction of speed causes the system to revert to a ten-cell structure plus a central stagnation zone when the line PQ is crossed. The change-over phenomena are now extremely weak effects and difficult to find, being covered by an extremely small range of speed. However, if sufficiently small changes in speed and long settling times are allowed, then reproducible results can still be obtained.

In common with the 6/8-cell mutation the loci of critical points is not as rounded as had been expected around the bifurcation point. Also, the cusp points upwards and its apex is about 4 % above the critical Reynolds number obtained from the infinite cylinder model, as compared with 5·5 % for the 6/8-cell cusp.

5. Conclusions

All of the experimental results presented here are broadly in accord with the qualitative ideas that were originally proposed by Benjamin concerning the primary-flow selection process in the Taylor experiment. The general class of behaviour exhibited by the cellular mutations does not change as the number of cells increases, although the exchanges examined here for the first time show several special and interesting features.

The direction in which the cusp in the (R, Γ)-diagram points cannot yet be predicted theoretically, and the experimental results suggest that this feature does not follow any simple rule. The downward-pointing cusp characterizing the eight- to ten-cell transition shows qualitative features in agreement with the results of Benjamin for two and four cells. Among the other cusps pointing upwards, the six–eight and ten–twelve mutations are qualitatively similar in the exchange region. The four–six mutation is notably more regular; however, the span of the hysteresis is an order of magnitude greater than all the others.

The strange behaviour of the six-cell state as a secondary mode may be tied somehow to the following observation. Taylor vortices apparently like to associate in pairs with an outward flowing jet between them. The six-cell state has a pair of such cells centrally located. At the point of collapse to the eight-cell mode, the symmetric conditions at the ends of the annulus try to enforce the creation of the extra pair of cells in the centre of a closed pair. This only happens when the aspect ratio is near the exchange region and the Reynolds number is small, giving a weaker outward flow. As the aspect ratio is increased, the six-cell mode apparently encounters an asymmetric collapse route, and the extra pair of cells are formed *between* existing pairs.

When the next highest odd number of pairs is involved (i.e. when the ten-cell state is a secondary mode), the collapse also occurs at small values of Reynolds number, just above the exchange region. Also, the rate of change of the locus of critical values with increasing aspect ratio is much smaller than for any of the even-paired modes. The coalescence of two pairs of cells to form one central pair on the collapse of an even- to odd-pair mode appears to be an easier process, however, and this is reflected in the regular features of the respective portions of the (Γ, R)-charts.

The 'pairing' of cells selectively in the special sense explained, which is a striking feature of the experimental observations reported here, will presumably have to be taken into account in any numerical solution of the problem.

The present results are believed to be novel, since nothing at all comparable is known other than Benjamin's findings for the two-cell to four-cell interchange. The experiments were delicate and time-consuming, but in the end they were found to be entirely repeatable and to reveal an orderly though complex pattern of behaviour. Accordingly, it would be welcome to see the results of an independent, perhaps improved, investigation of certain of the details discussed above. Already, however, one crucial conclusion supported consistently by all of these results is that *the presence of the ends is always crucial in determining what is observed in the rest of the flow.*

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